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ABSTRACT

This paper contains an elementary and short proof for the case that the underlying distribution function F is discrete, and then extends the result to the general F. In other proofs underlying iid sequences of random variables with continuous distributions are considered to be the "ideal" case. In this paper discretization of the underlying iid sequence is fundamental. Included are seven references and a list of titles of recent research reports. (Author/DC)

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# Ignatov's Theorem : A New and Short Proof

**Research Report 87-3**

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Ron Engelen
Paul Thommassen
Wim Vervaat

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Department of Education

Division of Educational Measurement and Data Analysis

University of Twente

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# Ignatov's Theorem: A New and Short Proof

Ron Engelen

Paul Tommassen

Wim Vervaat

Abstract

By Ignatov's Theorem the sets of values in an iid sequence that are the kth largest with their appearance (k=1,2,...) are supports of iid Poisson processes. The present paper contains an elementary and short proof for the case that the underlying distribution function F is discrete, and then extends the result to general F.

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## 1. Introduction.

In this paper we give a new proof of a result originally due to Ignatov (1977). Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed (iid) random variables and consider for each $k=1,2,\ldots$ the set of values in $X_1, X_2, \ldots$ which are the kth largest with their appearance (these values are called the kth order record values). Then the random sets of these values for $k=1,2,\ldots$, considered as point processes, are independent and identically distributed. Since the first formulation by Ignatov, many different proofs have appeared (cf. Deheuvels (1983), Goldie (1983), Goldie & Rogers (1984), Stam (1985)).

The point of view behind the proof in the present paper is strikingly different from that in the others. In the other papers, underlying iid sequences of random variables with continuous distributions are considered to be the "ideal" case, in which results allow the smoothest formulations, while the complementary case of distributions with atoms is seen as a nuisance, which only part of the authors handle explicitly. In the present approach, discretization of the underlying iid sequence is fundamental. Not only does this allow us to give a new proof, but it also provides us with insight why the results are true.

In fact, we will prove Ignatov's Theorem for the particular case that the common distribution of $X_1, X_2, \ldots$ is discrete in Section 2 without any preparation. The remainder of the paper only serves to

extend this result to the general case. Following Goldie (1983), we will formulate our results in the context of point processes, i.e., random integer-valued measures.

The general results are in Section 6, their proofs in Section 7. As a preparation, simplified Poisson processes are introduced in Section 3, processes of observations in Section 4, and processes of records in Section 5.

## 2. Discrete version of Ignatov's Theorem

For the proof of the discrete version of Ignatov's Theorem we need the following lemma:

Lemma 2.1: Let $X_1, X_2, \ldots$ be a sequence of real-valued iid random variables and let C be a Borel set in R such that $0 < P[X_1 \in C] < 1$. Set $\Delta_k := 1_{[X_k \in C]}$ and enumerate the k's with $\Delta_k = 1$ in increasing order by $(\tau_n)_{n=1}^{\infty}$ and the k's with $\Delta_k = 0$ by $(\sigma_n)_{n=1}^{\infty}$. Set $Y_n := X_{\tau_n}$ and $Z_n := X_{\sigma_n}$. Then the sequences $(\Delta_n)_{n=1}^{\infty}$, $(Y_n)_{n=1}^{\infty}$, $(Z_n)_{n=1}^{\infty}$ are independent, and each consists of iid random variables with distributions given by

$$P[\Delta_n = 1] = P[X_1 \in C],$$
$$P[Y_n \in A] = P[X_1 \in A \mid X_1 \in C],$$
$$P[Z_n \in B] = P[X_1 \in B \mid X_1 \notin C].$$

Proof: The joint distribution of $((\Delta_n)_{n=1}^{\infty}, (Y_n)_{n=1}^{\infty}, (Z_n)_{n=1}^{\infty})$ is determined by its values on the $\cap$-stable system of events $D := [\Delta_k=1$ for $k\in M$, $\Delta_k = 0$ for $k\in\{1,2,\ldots,n\}\setminus M$, $Y_k\in A_k$ for $k = 1,2,\ldots,\#M$, $Z_k\in B_k$ for $k=1,2,\ldots,n-\#M]$ with $n\in N$, $M\subset\{1,2,\ldots,n\}$, $A_k, B_k\in$ BorR. All claimed properties follow by evaluating $P(D)$ for $D\in D$.

The essence of our proof of Ignatov's Theorem is in the next lemma. Note that its conclusions are only non-trivial for atoms $x$ of the underlying distribution, i.e., for $x$ such that $P[X_1=x] > 0$. This explains why we will resort to discretization of the underlying distribution.

Lemma 2.2. (Discrete version of Ignatov's Theorem). Let $X_1, X_2, \ldots$ be a sequence of real-valued iid random variables, and define for $x\in R$

$$A_k(x) := \cup_{n=k}^{\infty} [X_n=x, \sum_{j=1}^{n} 1_{[X_j\geq x]}=k]$$
$$:= [x \text{ is a kth order record value}].$$

Then the events $A_k(x)$ are independent for $k\in N$ and $x\in R$, and

$$P(A_k(x)) = P[X_1=x | X_1\geq x].$$

Proof: Fix one $x\in R$ and apply Lemma 2.1 with $C = [x,\infty)$. Then we have $A_k(x) = [Y_k=x]$, so the $A_k(x)$ are independent for $k\in N$ with $P(A_k(x)) = P[X_1=x|X_1\geq x]$. For $\ell\in N$ and $y<x$ we have

$$A_\ell(y) = \cup_{n=1}^{\infty} \left[Z_n = y, \sum_{k=1}^{\sigma_n} \Delta_k + \sum_{j=1}^{n} 1_{[Z_j \geq y]} = \ell\right]$$

$$\in \sigma((\Delta_n)_{n=1}^{\infty}, (Z_n)_{n=1}^{\infty}).$$

The last $\sigma$-field is independent of $\sigma((Y_k))_{k=1}^{\infty})$, so the systems of random variables $(A_k(x))_{k\in N}$ and $(A_\ell(y))_{\ell\in N,\ y<x}$ are independent. By simple induction it follows that each finite subset of $(A_k(x))_{k\in N,\ x\in R}$ consist of independent events.

## 3. Simplified Poisson processes.

Let S be a locally compact space with countable base, M(S) the set of locally finite measures on BorS provided with the topology of vague convergence, and N(S) the subset of M(S) consisting of integer-valued measures. "Locally finite" here means finite on compact subsets of S. As usual (cf. Kallenberg (1976), Matthes et al. (1978)), a point process on S is an N(S)-valued random variable. Set $N^1(S) := \{\mu\in N(s) : \mu\{s\}\leq 1 \text{ for } s\in S\}$. Measures $\mu\in N^1(S)$ are called simple, and an $N^1(S)$-valued random variable is called a simple point process. Each $\mu\in N(S)$ can be transformed into a unique $\nu\in N^1(S)$ with the same support. We call $\nu$ the simplification of $\mu$ and write $\nu := \text{simp}\mu$. Let $\mu\in M(S)$. The point process $\pi$ on S is said to be Poisson with characteristic measure $\mu$ if $\pi(A)$ is Poisson distributed with mean $\mu(A)$ for each relatively compact $A\in BorS$ and the values of $\pi$ at disjoint sets are independent random variables.

We write $\pi=\pi_\mu$. Note that $\pi$ is simple iff $\mu$ is diffuse.

We will have to deal with $\text{simp}\pi_\mu$, the simplification of a Poisson process with characteristic measure $\mu$ (not necessarily diffuse). In general, the distribution of a simple point process $\xi$ is determined by $P[\xi(U) = 0]$ for $U$ in a sufficiently rich subclass of BorS. Specializing such a result to simplified Poisson processes, we have (cf. Goldie (1983)):

Proposition 3.1. Let $\mathbf{U}\subset\text{BorS}$ be a ring of relatively compact sets that generates BorS as a $\sigma$-ring. A simple point process $\xi$ is the simplification of a Poisson process $\pi_\mu$ with characteristic measure $\mu$ iff $P[\xi(U)=0] = \exp(-\mu(U))$ for all $U\in\mathbf{U}$.

## 4. Processes of observations

We now specify the basic space S for the point processes. It must serve the following purpose. Let $X_1,X_2,\ldots$ be a sequence of iid random variables in R. We want to represent this sequence by a simple point process in the plane with atoms $(T_n,X_n)$ for $n=1,2,\ldots$, where $(T_n)_{n=1}^\infty$ is some random increasing sequence in $[0,\infty)$. We are interested in large values in $(X_n)_{n=1}^\infty$, which will occur closely below $a := \sup\{x:P[X_1\le x]<1\}$, but not above $a$. So the natural choice for S is what we call the strip of observations

$$I := [0,\infty) \times (-\infty,a]$$

for some fixed (and minimal) $a \in (-\infty,\infty]$. Note that locally finite measures on BorI are north-west (NW) bounded, i.e., finite on the compact rectangles $[0,t] \times [x,a]$.

More specifically, let $(X_n)_{n=1}^{\infty}$ be a sequence of iid $(-\infty,a]$-valued random variables with distribution function F such that $F(x)<1$ for $x<a$. Let $(S_n)_{n=1}^{\infty}$ be a sequence of iid $(0,\infty)$-valued random variables with distribution function G, independent of $(X_n)_{n=1}^{\infty}$. Set $T_n := \sum_{k=1}^{n} S_k$. Then the point process of observations $Obs_{F,G}$ is defined by $Obs_{F,G} := \sum_{n=1}^{\infty} \delta_{(T_n,X_n)}$, where $\delta_{(t,x)}$ denotes Dirac measure at $(t,x) \in I$. The particular case $S_k=1$, $T_n=n$ wpl is denoted by $Obs_F$.

It is a central result of the theory of extremal processes that in many cases transformations of $Obs_F$ converge in distribution to Poisson processes on I with characteristic measure $\mu = Leb \times \nu$ for special $\nu$. We will write

$$\xi_\nu := \pi_{Leb \times \nu}.$$

For us the only relevant properties are that $\nu$ is locally finite on $(-\infty,a]$ (i.e., $\nu[x,a] < \infty$ for $-\infty<x\leq a$), so $Leb \times \nu$ is locally finite on I, and that a is minimal: $\nu[x,a] > 0$ for $x<a$. In the particular case that $\nu$ is finite, the Poisson process $\xi_\nu$ can be regarded as a process of observations (Goldie (1983)):

Proposition 4.1. If $\nu(-\infty,a] < \infty$, then $\xi_\nu \overset{\mathcal{D}}{=} Obs_{F,G}$, where $F(x) = \nu(-\infty,x]/\nu(-\infty,a]$ for $x \leq a$ and $G(t) = 1-\exp(-t\nu(-\infty,a])$ for $t \geq 0$.

When $\nu$ is not finite, the proposition still applies to $\nu_y$ with $d\nu_y := 1_{[y,a]}d\nu$ for $-\infty < y < a$.

## 5. Observations and records

Let $\mu \in N(I)$ be fixed. The rank of $(t,x) \in I$ with respect to $\mu$ is defined by

$$r(t,x) := \mu([0,t]\times[x,a]) ,$$

so r is in fact the NW distribution function of $\mu$. The terminology is motivated by the following. The rank of $(T_n,X_n)$ in $Obs_{F,G}$ is equal to $\#\{k\leq n: X_k \geq X_n\}$. Note that $r(T_n,X_n)$ does not depend on $(T_n)_{n=1}^{\infty}$, so that the distribution of the ranks of the atoms of $Obs_{F,G}$ does not depend on G.

We now rearrange all information about the ranks of point processes on the strip I into new point processes. Let $\mu \in N(I)$ be fixed. Besides the rank r(t,x) we consider the record of (t,x):

$$rec(t,x) := (r(t,x),x).$$

Putting aside for the moment the question of local finiteness, we now define the following random integer-valued measures on various spaces S:

the kth record values (k=1,2,..)

$$RV_k\mu := \sum_{(t,x)\ \in \mathrm{supp}\mu, r(t,x)=k} \delta_x,\ S=(-\infty,a]\ ,$$

and the records:

$$\mathrm{Rec}\mu := \sum_{(t,x)\ \in \mathrm{supp}\mu} \delta_{(r(t,x),x)}\ ,\ S=N\times(-\infty,a]\ ,$$

so that $\mathrm{Rec}\mu$ is the combination of the random measures $RV_k\mu$ acting on (the Borel sets of) the kth vertical $\{k\}\times(-\infty,a]$ in the collection of verticals $N\times(-\infty,a]$.

Here are the local finiteness properties of these processes (see Goldie (1983) for proofs):

1. $RV_k$ is locally finite on $(-\infty,a)$ (so there is a largest kth order record value);
2. $RV_k$ is locally finite on $(-\infty,a]$ iff $\mu([0,\infty)\times\{a\}) \geq k$ or $\mu([0,\infty)\times[x,a)) < k$.

Furthermore, $RV_k\mu$ is simple even if $\mu$ is not simple.

Considering $\mu = \mathrm{Obs}_{F,G}$ and $\mu=\xi_\nu$ as in the previous section, we see that possibility 2 is excluded if

(3.1.a) $F(a)=1$, $F(x) < 1$ for $x < a$,

(3.1.b) $\nu\{a\}=0$, $\nu([x,a]) > 0$ for $x < a$,

in which cases $\mu([0,\infty)\times\{a\}) = 0$ wpl and $\mu$ is north-east unbounded: $\mu([0,\infty)\times(x,a)) = \infty$ for all $x<a$, wpl. Henceforth we will assume (3.1). As a consequence we may and will regard $\mathrm{Rec}\mu$ and $RV_k\mu$ as simple point processes on $N\times(-\infty,a)$, $(-\infty,a)$ respectively, with

finite values on the compact rectangles $\{0,1,\ldots,n\}\times[x,y]$, $[x,y]$ respectively, for $-\infty < x < y < a$.

## 6. Ignatov's Theorem

The theorems apply to $Obs_{F,G}$ and $\xi_\nu$ as defined in Section 4. The hypotheses (3.1) are assumed throughout.

Theorem 6.1. The point processes $RV_k Obs_{F,G}$ are iid. Each of them is the simplification of a Poisson process on $(-\infty,a)$ with characteristic measure $\eta$ given by $\eta(-\infty,x] = -\log(1-F(x))$ for all $x < a$.

Theorem 6.2. The point proceses $RV_k \xi_\nu$ are iid. Each of them is the simplification of a Poisson process on $(-\infty,a)$ with characteristic measure $\eta$ given by $\eta(x,y] = \log(\nu(x,a]/\nu(y,a])$ for $-\infty<x<y<a$.

Both theorems can be reformulated succintly, with # denoting counting measure.

Theorem 6.3. The point proceses $RecObs_{F,G}$ and $Rec\xi_\nu$ on $N\times(-\infty,a)$ are simplifications of Poisson processes with characteristic measure $\mu = \#\times\eta$ with $\eta$ as in Theorems 6.1 and 6.2 .

If F is continuous or $\nu$ is diffuse, then the Poisson processes in the conclusions of the theorems have diffuse characteristic

measures, so are already simple themselves. If F in Theorem 6.1 is discrete, then Theorem 6.1 specializes to Lemma 2.2: $\eta$ is discrete; $RV_k Obs_{F,G}$ can have atoms only at atoms x of $\eta$ and they occur independently with probability $1-\exp(-\eta(x)) = 1-(1-F(x))/(1-F(x-)) = P[X_1=x \mid X_1 \geq x]$.

## 7. Proofs

Theorem 6.2 follows from Theorem 6.1 by Proposition 4.1 in case $\nu$ is finite. In the complementary case, consider $\nu_y$ with $d\nu_y := 1_{[y,a]} d\nu$ for $-\infty<y<a$, and note that $dRV_k\xi_{\nu_y} = 1_{[y,a]} dRV_k\xi_\nu$. Then applying Theorem 6.1 and Proposition 4.1 once more, we see that for all $y<a$ the restrictions of $RV_k\xi_\nu$ to $[y,a]$ behave as they should according to Theorem 6.2, from which the theorem follows. So Theorem 6.2 follows from Theorem 6.1.

As Theorem 6.1 reduces for discrete F to Lemma 2.2, which has been proved already, it remains to prove Theorem 6.1 for general F, and special G as $Obs_{F,G}$ does not depend on G. We will do this as follows. We view the image of a NW piece of $Obs_F$ as projected on a TV screen with finitely many horizontal lines, apply Lemma 2.2 to the TV picture and then let increase the number of horizontal lines on the screen.

Proof: We prove Theorem 6.1 in the formulation of Theorem 6.3 for $Obs_F$. We are going to apply Proposition 3.1 to $RecObs_F$, a simple

point process in $N\times(-\infty,a)$.

Let $\mathbf{V}$ be the collection $\{(u,v]\,|\,u<v,\ u,v\in(-\infty,a)\}$ and let $\mathbf{U}$ be the ring in $N\times(-\infty,a)$ consisting of all unions of finitely many sets $\{j\}\times V$ with $j\in N$ and $V\in\mathbf{V}$. Then $\mathbf{U}$ satisfies the conditions of Proposition 3.1, and it remains to prove

$$(1)\qquad P[\mathrm{RecObs}_F(U) = 0] = \exp(-\#\times\eta(U)) \text{ for } U \in \mathbf{U}.$$

Fix a non-empty $U\in\mathbf{U}$. It can be written as $U = \cup_{j=1}^{k} \{j\}\times W_j$ for some $k\in N$ and $W_j$ the union of finitely many elements of $\mathbf{V}$ for $j=1,2,\ldots,k$. Then (1) is equivalent to

$$(2)\qquad P \cap_{j=1}^{k} [\mathrm{RV}_j\mathrm{Obs}_F(W_j) = 0] = \exp(-\textstyle\sum_{j=1}^{k} \eta(W_j))$$

Let $D_n$ for $n=1,2,\ldots$ be finite subsets of $(-\infty,a)$ such that $D_1$ contains the boundary points of all $W_j$, $D_n \subset D_{n+1}$ for $n\geq 1$ and $\cup_{n=1}^{\infty} D_n$ is dense in $(-\infty,a)$. Let $T_n$ be the transformation that lifts the atoms $(m,x)$ of point processes in $N\times(-\infty,a)$ to the nearest higher level in $D_n$, to $(m,\inf([x,a]\cap D_n))$. Then $T_n\mathrm{Obs}_F \overset{d}{=} \mathrm{Obs}_{F_n}$, where $F_n(x) := F(\inf([x,a]\cap D_n))$. Note that $F_n$ is discrete, so (1) and (2) hold true with $F$ and $\eta$ replaced by $F_n$ and the corresponding $\eta_n$. Furthermore, $F_n(x)=F(x)$ for $x\in D_n$, in particular for $x$ at the boundary of $W_j$, so $\eta_n(W_j) = \eta(W_j)$ for $j=1,2,\ldots,k$. We have arrived at

$$(3)\qquad P\,[\mathrm{Rec}(T_n\mathrm{Obs}_F)(U)=0] = P[\mathrm{RecObs}_{F_n}(U)=0] = \exp(-(\#\times\eta_n)(U)) = \exp(-(\#\times\eta)(U))$$

It may happen that

$$(4) \qquad \mathrm{Rec}(T_n \mathrm{Obs}_F) = T_n(\mathrm{RecObs}_F) \text{ on } \{1,2,\ldots,k\}\times \cup_{j=1}^{k} W_j.$$

A sufficient condition for this is that T does not move atoms of $\mathrm{Obs}_F$ in $Q := \{1,2,\ldots,\tau_k\}\times(-\infty,a)$ from different levels to the same level, where $\tau_k$ is as in Lemma 2.1 with $C=[\sup \cup_{j=1}^{k} W_j,\infty)$. A sufficient condition for the latter is that all atoms of $\mathrm{Obs}_F$ in Q at different levels are seperated by levels from $D_n$. This is the case for all sufficiently large n (depending on the realization of $\mathrm{Obs}_F$), as $\mathrm{Obs}_F$ has finitely many points in Q and $D_n$ increases to a dense set in $(-\infty,a)$. So we have (4) for all these n, whence

$$\mathrm{Rec}(T_n \mathrm{Obs}_F)(U) = 0 \text{ iff } T_n(\mathrm{RecObs}_F)(U) = 0$$
$$\text{iff } \mathrm{RecObs}_F(U) = 0$$

for the same n.

We have proved that the events $[\mathrm{Rec}(T_n \mathrm{Obs}_F)(U)=0]$ converge to the event $[\mathrm{RecObs}_F(U)=0]$ as $n \to \infty$, and so do the corresponding probabilities. This combined with (3) gives (1).

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## Authors' Note

The addresses of the co-authors are: Paul Thommassen, PTT Centrale Directie, Prinses Beatrixlaan 16, 2595 AL Den Haag and Wim Vervaat, Mathematisch Instituut, Katholieke Universiteit, Toernooiveld 5, 6525 ED Nijmegen.

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